Rocket motion during vertical powered and unpowered flight

Jørgen Franck, Danish Amateur Rocket Club

The equation of motion of a rocket can be obtained from the law of conservation of momentum. Let the mass of the rocket at any given instant be m, and let its speed be v relative to some fixed coordinate system. If material is shot out of the rocket motor with an exhaust velocity v_{ex} relative to the rocket, the velocity of the exhaust relative to the fixed coordinate system is $v - v_{ex}$. If an external force F also acts on the rocket, then the linear momentum theorem reads in this case:

$$\frac{d}{dt}(mv) - \left(v - v_{ex}\right)\frac{dm}{dt} = -F$$

The first term is the time rate of change of momentum of the rocket. The second term represents the rate at which momentum is appearing in the rocket exhaust, where $-dm/dt$ is the rate at which matter is being exhausted. If we fix our attention on the rocket at any moment, we must remember that at a time dt later this system will comprise the rocket plus the material exhausted from the rocket during that time, and both must be considered in computing the change in momentum. The force F represent the air resistance and the gravitational force, we get

$$m\frac{dv}{dt} + v_{ex}\frac{dm}{dt} = -mg - kv^2 \Rightarrow$$

$$m\frac{dv}{dt} + kv^2 = -v_{ex}\frac{dm}{dt} - mg$$

In the following we assume that g, k and v_{ex} are constant, and the mass of the rocket is a linear function of time

$$m = m_0 + \frac{dm}{dt}t = m_0 + bt$$

where m_0 is the initial mass of the rocket. Further is $dm/dt < 0$ and constant. Remember that $-dm/dt$ is the rate at which matter is being exhausted. The constant k is equal to

$$k = \frac{1}{2}rAc_D$$

Now we transform the differential equation of motion by a change of dependent variable

$$v = \frac{m}{ky}\frac{dy}{dt}$$

$$\frac{dv}{dt} = \frac{1}{ky^2}\left\{by\frac{dy}{dt} + my\frac{d^2y}{dt^2} - m\left(\frac{dy}{dt}\right)^2\right\} = \frac{b}{ky}\frac{dy}{dt} + \frac{m}{ky}\frac{d^2y}{dt^2} - \frac{m}{ky^2}\left(\frac{dy}{dt}\right)^2$$

and it takes the form

$$\frac{bm}{ky}\frac{dy}{dt} + \frac{m^2}{ky}\frac{d^2y}{dt^2} - \frac{m^2}{ky^2}\left(\frac{dy}{dt}\right)^2 + \frac{m^2}{ky^2}\left(\frac{dy}{dt}\right)^2 = -v_{ex}b - mg \Rightarrow$$

$$m^2\frac{d^2y}{dt^2} + bm\frac{dy}{dt} + k(v_{ex}b + mg)y = 0$$

We transform again

$$y(t) = u(z)\big|_{z=m_0+bt}$$

$$\frac{d}{dt} = b\frac{d}{dz}, \qquad \frac{d^2}{dt^2} = b^2\frac{d^2}{dz^2}$$

and get

$$b^2z^2\frac{d^2u}{dz^2} + b^2z\frac{du}{dz} + k(gz + v_{ex}b)u = 0$$

The equation may be written in the form

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + (az + b)u = 0$$

where

$$a = \frac{gk}{b^2}, \qquad b = \frac{v_{ex}k}{b} < 0$$

This equation can be transformed into Bessel's equation and has the solution

$$u = C_1J_g(2\sqrt{az}) + C_2J_{-g}(2\sqrt{az}), \qquad g \text{ nonintegral}$$

where J_g is a Bessel function of the first kind and the order is equal to

$$g = 2\sqrt{-b}$$

The boundary conditions are

$$t = 0: \qquad v = 0 \Rightarrow$$

$$z = m_0: \qquad \frac{du}{dz} = 0$$

The derivative of u is

$$\frac{du}{dz} = \frac{C_1}{2}\sqrt{\frac{a}{z}}\left\{J_{g-1}(2\sqrt{az}) - J_{g+1}(2\sqrt{az})\right\} + \frac{C_2}{2}\sqrt{\frac{a}{z}}\left\{J_{-g-1}(2\sqrt{az}) - J_{-g+1}(2\sqrt{az})\right\}$$

From $du/dz = 0$, we get

$$C_2 = -C_1 \frac{J_{g-1}(h) - J_{g+1}(h)}{J_{-g-1}(h) - J_{-g+1}(h)} = \Theta C_1$$

where

$$h = 2\sqrt{am_0}$$

Now u and du/dz becomes

$$u = C_1\left\{J_g(2\sqrt{az}) + \Theta J_{-g}(2\sqrt{az})\right\}$$

$$\frac{du}{dz} = \frac{C_1}{2}\sqrt{\frac{a}{z}}\left\{J_{g-1}(2\sqrt{az}) - J_{g+1}(2\sqrt{az}) + \Theta\left[J_{-g-1}(2\sqrt{az}) - J_{-g+1}(2\sqrt{az})\right]\right\}$$

Finally the burnout velocity can be obtained

$$v_b = \frac{m}{ky}\frac{dy}{dt} \Rightarrow$$

$$v_b = \frac{zb}{ku}\frac{du}{dz} \Rightarrow$$

$$v_b = \frac{bm}{4k}\frac{J_{g-1}(m) - J_{g+1}(m) + \Theta\left\{J_{-g-1}(m) - J_{-g+1}(m)\right\}}{J_g(m) + \Theta J_{-g}(m)}$$

where

$$m = 2\sqrt{az}$$

The altitude at burnout is determined by integrating the expression for the velocity with respect to time as follows

$$v = \frac{zb}{ku}\frac{du}{dz} \Rightarrow$$

$$\frac{ds}{dt} = \frac{zb}{ku}\frac{du}{dz} \Rightarrow$$

$$\int_0^{s_b} ds = \frac{b}{k}\int_0^{t_b} \frac{z}{u}\frac{du}{dz} dt \Rightarrow$$

$$s_b = \frac{1}{k}\int_{m_0}^{m_0+bt_b} \frac{z}{u}\frac{du}{dz} dz \Rightarrow$$

$$s_b = \frac{1}{k}\int_{m_0}^{m_0+bt_b} \frac{z}{u} du \Rightarrow$$

$$s_b = \frac{1}{k}\left\{\left[z\log(u)\right]_{m_0}^{m_0+bt_b} - \int_{m_0}^{m_0+bt_b} \log(u) dz\right\}$$

Unfortunately the altitude at burnout cannot be solved in closed form as the burnout velocity. After burnout of the rocket the vertical differential equation of motion becomes

$$m_b \frac{dv}{dt} = -m_b g - kv^2$$

where m_b is the burnout mass of the rocket. The solutions for the coasting phase are well-known, but are listed below for a generel survey of the problem. This relation may be rearranged to solve for t as a function of v, and one then get the coasting time

$$-\frac{m_b}{k}\int_{v_b}^{0} \frac{dv}{\frac{m_b g}{k} + v^2} = \int_0^{t_c} dt \Rightarrow$$

$$t_c = \sqrt{\frac{m_b}{gk}} \arctan\left[v_b \sqrt{\frac{k}{m_b g}}\right]$$

The equation can also be solved for the coasted altitude increment by introducing the following transformation of variables

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}$$

When integrated, it yields the coasted altitude increment as

$$m_b v \frac{dv}{ds} = -m_b g - kv^2 \Rightarrow$$

$$-\frac{m_b}{k}\int_{v_b}^{0} \frac{v dv}{\frac{m_b g}{k} + v^2} = \int_0^{s_c} dt \Rightarrow$$

$$s_c = \frac{m_b}{2k} \ln\left[\frac{kv_b^2}{m_b g} + 1\right]$$

References:
1. Mandell, G. etal.:Topics inAdvanced ModelRocketry, The MIT Press
2. Spiegel, M.:AdvancedMathematics,Schaum'sOuline Series.
3. Spiegel, M.:TheoreticalMechanics,Schaum'sOuline Series.