The N-body Problem

Jørgen Franck, Danish Amateur Rocket Club

During the great plague in 1666 the English universities closed. Isaac Newton was then an undergraduate in his early twenties at Cambridge. He returned to his mum's farm in Woolsthorpe, Lincolnshire to remain out of danger. Here he made an outstanding contribution to the gravitational interaction between two bodies, either planets or small particles. Using Kepler's laws, he derived the law of universal gravitation. The law may be stated as follows.

Every body in the universe attracts every other body by a force proportional to the product of their masses and inversely proportional to the square of the distance between them.

Mathematically the law of universal gravitation may be expressed in the following manner. If m_1 and m_2 are the masses of two bodies separated by a distance r between the bodies centers, the force of attraction F is

$$F = G\frac{m_1 m_2}{r^2}$$

The constant G is called the constant of gravitation and its value is

$$G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$$

Actually Newton postponed the publication of the discovery in nearly twenty years, because he had difficulties in solving a particular problem in integral calculus, which was conclusive for the whole theory. Strictly Newton's law of universal gravitation only apply for point masses, but planets have finite size which may introduce some geometrical factor. This problem teased Newton, but finally he could prove that a solid homogeneous sphere produces a gravitational field identical to those of a particle of the same mass located in the center of the sphere. Actually the result still holds true when the sphere has its mass distributed with spherical symmetry. Finally the discovery were published in 1687, when it appeared as a chapter in his famous work *Philosophiae Naturalis Principia Mathematica* .

Newton's law of universal gravitation is the cornerstone in the n-body problem, which may be stated in the following manner.

Consider n bodies distributed in space. Assume that all masses, positions and velocities are known and all the bodies experiences a gravitational attraction according to Newton's law of universal gravitation. What are the position and velocity of each body at any time?

Newton solved the problem completely for n = 2. The complexity of the problem grows considerably for n = 3 and is the famous three-body problem. Since the eighteenth century the three-body problem is considered as one of the most difficult to solve in mathematics. For the solution of the three-body problem King Oscar II of Sweden offered a prize and a gold medal. Poincaré got this price in 1889 for his discussion of the problem, but up to date nobody have solved the n-body problem for n equal or bigger than three.

Now we will use a computer and an algorithm to solve the n-body problem numerical. This illustrates how a complex mathematical problem can be expressed by an algorithm in a computer. Further the numerical solution can be obtained, due to the computers capability to perform repeated execution of an algorithm with very small time intervals. We use the constant acceleration approximation expressed in vector form as

$$\vec{R} = \vec{R}_0 + \vec{V}_0 \Delta t + ½ \vec{A}_0 \Delta t^2$$

$$\vec{V} = \vec{V}_0 + \vec{A}_0 \Delta t$$

where the acceleration $\vec{A}_0$ and velocity $\vec{V}_0$ is assumed constant in a small time interval $\Delta t = t - t_0$. Further $\vec{R}_0$ gives the position at time t_0. Finally $\vec{R}$ and $\vec{V}$ are respectively the position and velocity at time t. In the following we will use the notation

$A(J,I)$: Acceleration of the J'th planet in the I'th dimension.
$V(J,I)$: Velocity of the J'th planet in the I'th dimension.
$R(J,I)$: Position of the J'th planet in the I'th dimension.

The above quantities are measured relative to a well-defined frame of reference. The form of Newton's law of universal gravitation is not very usefull when the planets motion are described relative to a frame of reference, since the force of attraction is expressed as a function of distance rather than coordinates. We will rewrite Newton's law of universal gravitation to express the acceleration acting on the J'th planet in the I'th direction, we get

$$A_0(J,I) = A(J,I) + \sum_{K \neq J}^{N} \frac{G * M(K) * (R_0(K,I) - R_0(J,I))}{D(K,J)}$$

where

$$D(K,J) = D(J,K) = \sum_{I} \left| (R_0(K,I) - R_0(J,I))^3 \right|$$

The first term $A(J,I)$ is the acceleration the planet got from the previous calculation and the second term is the acceleration increment due to the other planets. We can of course not include the term $K = J$ in the sum, because the actual planet does not influence on itself. Further we get division by zero when trying to calculate the second term in this case.

Below is the algorithm implemented in the BASIC programming language. The input and output section is omitted. Further the algorithm only solve the n-body problem in two dimensions.

```
10    REM N-BODY PROBLEM
20    REM INPUT
.
.
.
230   REM CALCULATION
240   FOR J = 1 TO N
250     FOR K = 1 TO J - 1
260       D = (R(K,1) - R(J,1)) ^ 2 + (R(K,2) - R(J,2)) ^ 2
270       D(J,K) = D * SQR(D)
280       D(K,J) = D(J,K)
290     NEXT K
300   NEXT J
310   FOR J = 1 TO N
320     FOR I = 1 TO 2
330       FOR K = 1 TO N
340         IF K = J THEN GOTO 360
350         A(J,I) = A(J,I) + G * M(K) * (R(K,I) - R(J,I)) / D(K,J)
360       NEXT K
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370       R(J,I) = R(J,I) + V(J,I) * T0 + 0.5 * A(J,I) * T0 * T0
380       V(J,I) = V(J,I) + A(J,I) * T0
390     NETX I
400   NEXT J
410   T = T + T0
420   REM OUTPUT
.
.
.
530   GOTO 240
```

The input section of the program first read the time interval T0 and the number of planets N. Then the individual planets masses, coordinates and velocities are read. Last the entered planet positions are plotted on the computer screen. Remember to define the constant of gravitation G and set the duration T = 0.

The calculation section first determine the D matrix for all the planets (line no. 240-300). When this is done, the acceleration, coordinates and velocities are determined for the N planets (line no. 310-400). The duration T is calculated by adding the time interval Δt (line no. 410).

The output section plot the planets new positions on the computer screen.

Finally the program jump back (line no. 530) and calculates the D matrix again based on the new coordinates. Hereby the program run in an infinite loop, calculating new accelerations, velocities and coordinates for all the planets for a given time interval Δt.

The program solve the n-body problem for two dimensions, because it then is easy to plot the planet positions on the computer screen. But it is easy to expand the calculation to three dimensions. Add (R(K,3) - R(J,3)) ^ 2 in line no. 260 and finally change 2 to 3 in line no. 320.

If the distance between the calculated planet position increase on the computer screen the velocity of the planet increase too, because the time interval Δt between the positions are constant.

The algorithm does not solve the n-body problem very accurate, due to the simple numerical method used.