Beregning af areal, volumen, massemidtpunkt og inertimomenter for en klasse af omdrejningslegemer med cirkelbuegeometri

af

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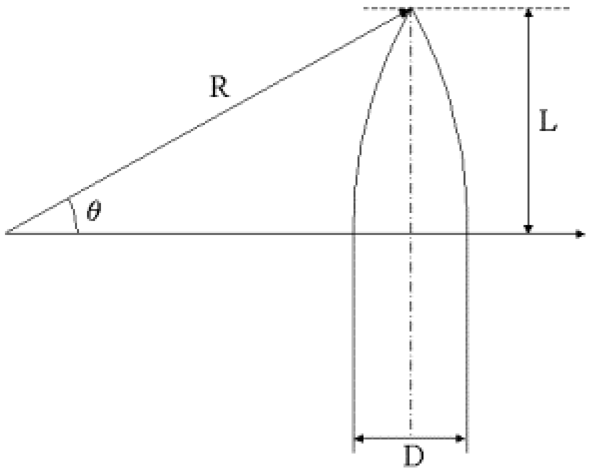
Introduktion

Denne rapport gennemgår beregningen af areal, volumen, massemidtpunkt samt inertimomenter for en klasse af omdrejningslegemer med cirkelbuegeometri. Et af tilfældene er en tangent ogive som ofte anvendes som raketspidsen på raketter.



Af ovenstående figur kan radius R beregnes som følger

$$R = \frac{D}{2} + R\cos\theta$$

Da $L = R\sin\theta$ og $\cos^2\theta + \sin^2\theta = 1$, får vi

$$\cos\theta = \sqrt{1 - \left(\frac{L}{R}\right)^2}$$

Sammenfattes de to udtryk fås

$$R = \frac{D}{4} + \frac{L^2}{D}$$

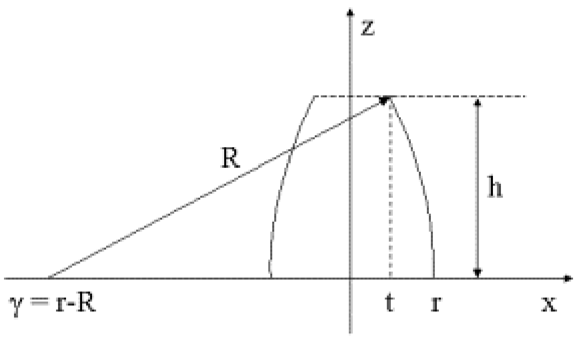
Ved omskrivning kan længe-diameter forholdet udtrykkes

$$\frac{L}{D} = \sqrt{\frac{R}{D} - \frac{1}{4}}$$

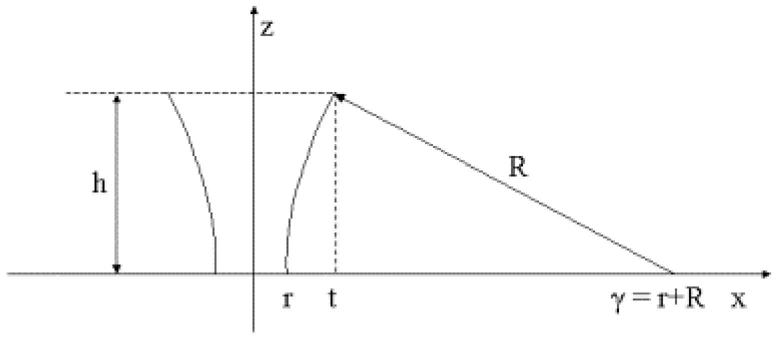
Areal

Arealet af den krumme overflade for et omdrejningslegeme med cirkelbuegeometri beregnes for de to nedenstående tilfælde.

Tilfælde I:



Tilfælde II:



Cirklens ligning er nu i de to tilfælde

$$z^2 + (x-\gamma)^2 = R^2$$

Parameterfremstillingen i (x,z)-planen er

$$x = f(u) = u + \gamma \,, \qquad f'(u) = 1$$

$$z = g(u) = \sqrt{R^2 - u^2}\,, \qquad g'(u) = \frac{-u}{\sqrt{R^2 - u^2}}$$

hvor der gælder følgende:

Tilfælde I: $t - \gamma \le u \le R$ og $0 \le t < r$

Tilfælde II: $-R \le u \le t - \gamma$ og $0 \le r < t$

Følgende dimensionsløse størrelser indføres

$$\lambda = \frac{h}{R} \quad \text{og} \quad \begin{cases} \mu = 1 - \dfrac{r}{R} & \text{for} \quad 0 \le t < r \\ \mu = 1 + \dfrac{r}{R} & \text{for} \quad 0 \le r < t \end{cases}$$

Drejes kurven én omgang omkring z-aksen fremkommer der en flade. Arealet af denne omdrejningsflade kan beregnes ved følgende formel

$$A = 2\pi \int_\alpha^\beta f(u)\sqrt{(f'(u))^2 + (g'(u))^2}\,du$$

Indsættes parameterfremstillingen fås

$$A = 2\pi \int_{\alpha}^{\beta} (u+\gamma)\sqrt{1-\frac{u^2}{R^2-u^2}}du \Rightarrow$$

$$A = 2\pi \int_{\alpha}^{\beta} (u+\gamma)\sqrt{\frac{R^2}{R^2-u^2}}du \Rightarrow$$

$$A = 2\pi R \int_{\alpha}^{\beta} \frac{u+\gamma}{R^2-u^2}du \Rightarrow$$

$$A = 2\pi R\left[-\sqrt{R^2-u^2}+\gamma\ \mathrm{Arcsin}\ \frac{u}{R}\right]_{\alpha}^{\beta}$$

Indsættes grænserne fra tilfælde I fås

$$A = 2\pi R\left[-\sqrt{R^2-u^2}+\gamma\ \mathrm{Arcsin}\ \frac{u}{R}\right]_{t-\gamma}^{R} \Rightarrow$$

$$A = 2\pi R\left(\sqrt{R^2-(t-\gamma)^2}+\gamma\left(\frac{\pi}{2}-\mathrm{Arcsin}\ \frac{t-\gamma}{R}\right)\right)$$

Indsættes grænserne fra tilfælde II fås

$$A = 2\pi R\left[-\sqrt{R^2-u^2}+\gamma\ \mathrm{Arcsin}\ \frac{u}{R}\right]_{-R}^{t-\gamma} \Rightarrow$$

$$A = 2\pi R\left(-\sqrt{R^2-(t-\gamma)^2}-\gamma\left(-\frac{\pi}{2}-\mathrm{Arcsin}\ \frac{t-\gamma}{R}\right)\right)$$

Ved geometribetragtninger kan følgende udledes for tilfældene

I: $R^2 = h^2+(t-\gamma)^2 \Leftrightarrow h=\sqrt{R^2-(t-\gamma)^2} \Leftrightarrow t-\gamma=\sqrt{R^2-h^2}$

II: $R^2 = h^2+(\gamma-t)^2 \Leftrightarrow h=\sqrt{R^2-(\gamma-t)^2} \Leftrightarrow \gamma-t=\sqrt{R^2-h^2}$

Ved indsættelse fås nu

I: $$A = 2\pi R\left(h+(r-R)\left(-\frac{\pi}{2}-\mathrm{Arcsin}\sqrt{1-\left(\frac{h}{R}\right)^2}\right)\right)$$

II: $$A = 2\pi R\left(-h+(r+R)\left(\frac{\pi}{2}+\mathrm{Arcsin}\left(-\sqrt{1-\left(\frac{h}{R}\right)^2}\right)\right)\right)$$

De to arealformler kan nu sammenfattes til

$$A = 2\pi R\left(\pm h + (r \mp R)\left(\frac{\pi}{2} - \mathrm{Arcsin}\sqrt{1-\left(\frac{\mathrm{h}}{\mathrm{R}}\right)^2}\right)\right) \Rightarrow$$

$$A = 2\pi R\left(\pm h - (\pm R - r)\,\mathrm{Arccos}\sqrt{1-\left(\frac{\mathrm{h}}{\mathrm{R}}\right)^2}\right) \Rightarrow$$

$$A = 2\pi R^2\left(\pm\frac{h}{R} - \left(\pm 1 - \frac{r}{R}\right)\mathrm{Arcsin}\,\frac{\mathrm{h}}{\mathrm{R}}\right) \Rightarrow$$

$$A = 2\pi R^2\left(\pm\lambda - (\pm\mu)\,\mathrm{Arcsin}\,\lambda\right) \quad , \quad \begin{cases} + \text{ i tilfælde I, } 0 \le t < r \\ - \text{ i tilfælde II, } 0 \le r < t \end{cases}$$

hvor

$$\mu = 1 \mp \frac{r}{R} \quad , \quad \begin{cases} - \text{ i tilfælde I, } 0 \le t < r \\ + \text{ i tilfælde II, } 0 \le r < t \end{cases}$$

$$\lambda = \frac{h}{R}$$

Volumen

Til beregning af volumen omskrives cirklens ligning for de to tilfælde til

$$x = \gamma \pm \sqrt{R^2 - z^2} \, , \qquad \begin{cases} + \text{ i tilfælde I, } 0 \le t < r \\ - \text{ i tilfælde II, } 0 \le r < t \end{cases}$$

Volumen af omdrejningslegemet er

$$V = \pi\int_0^h x^2 dz \Rightarrow$$

$$V = \pi\int_0^h \left(\gamma \pm \sqrt{R^2 - z^2}\right)^2 dz$$

Indføres følgende substitution

$$t = \frac{z}{R} \quad , \quad dz = R dt$$

Vi får nu

$$V = \pi R^3 \int_0^\lambda \left(\frac{\gamma}{R} \pm \sqrt{1 - t^2}\right)^2 dt \Rightarrow$$

$$V = \pi R^3 \int_0^\lambda \left(\left(\frac{\gamma}{R} \right)^2 + (1 - t^2) \pm \frac{2\gamma}{R} \sqrt{1 - t^2} \right) dt \Rightarrow$$

$$V = \pi R^3 \left[\left(\frac{\gamma}{R} \right)^2 t + t - \frac{t^3}{3} \pm \frac{2\gamma}{R} \left(\frac{t\sqrt{1 - t^2}}{2} + \frac{1}{2} \text{Arcsin } t \right) \right]_0^\lambda \Rightarrow$$

$$V = \pi R^3 \left(\left(\frac{\gamma}{R} \right)^2 \lambda + \lambda - \frac{\lambda^3}{3} \pm \frac{2\gamma}{R} \left(\frac{\lambda \sqrt{1 - \lambda^2}}{2} + \frac{1}{2} \text{Arcsin } \lambda \right) \right) \Rightarrow$$

$$V = \pi R^3 \left(\lambda \left(1 + \left(\frac{\gamma}{R} \right)^2 - \frac{\lambda^2}{3} \pm \frac{\gamma}{R} \sqrt{1 - \lambda^2} \right) \pm \frac{\gamma}{R} \text{Arcsin } \lambda \right) \Rightarrow$$

$$V = \pi R^3 \left(\lambda \left(1 + \left(\frac{r \mp R}{R} \right)^2 - \frac{\lambda^2}{3} \pm \frac{r \mp R}{R} \sqrt{1 - \lambda^2} \right) \pm \frac{r \mp R}{R} \text{Arcsin } \lambda \right) \Rightarrow$$

$$V = \pi R^3 \left(\lambda \left(1 + \left(\frac{r}{R} \mp 1 \right)^2 - \frac{\lambda^2}{3} \pm \left(\frac{r}{R} \mp 1 \right) \sqrt{1 - \lambda^2} \right) \pm \left(\frac{r}{R} \mp 1 \right) \text{Arcsin } \lambda \right) \Rightarrow$$

$$V = \pi R^3 \left(\lambda \left(1 + \left(1 \mp \frac{r}{R} \right)^2 - \frac{\lambda^2}{3} - \left(1 \mp \frac{r}{R} \right) \sqrt{1 - \lambda^2} \right) - \left(1 \mp \frac{r}{R} \right) \text{Arcsin } \lambda \right) \Rightarrow$$

$$V = \pi R^3 \left(\lambda \left(1 + \mu^2 - \frac{\lambda^2}{3} - \mu \sqrt{1 - \lambda^2} \right) - \mu \text{ Arcsin } \lambda \right)$$

hvor

$$\mu = 1 \mp \frac{r}{R} \quad , \quad \begin{cases} - \text{ i tilfælde I}, & 0 \le \text{t} < \text{r} \\ + \text{ i tilfælde II}, & 0 \le \text{r} < \text{t} \end{cases}$$

$$\lambda = \frac{h}{R}$$

Massemidtpunkt

Af symmetrigrunde er massemidtpunktet placeret på z-aksen. Massemidtpunktets z-koordinat kan findes ved følgende formel

$$\Psi = \frac{\int z\rho(x,y,z)dV}{\int \rho(x,y,z)dV}$$

Massetætheden °*(x,y,z)*antages konstant hvorved den kan bortforkortes. Legemet kaldes homogent når °*(x,y,z)*er konstant. Integralet i nævneren er nu lig volumen af det aktuelle omdrejningslegeme. Integralet i tælleren kan findes som følger

$$\Phi = \int z dV$$

Indføres

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$
$$dV = r d\phi dr dz$$

Vi får nu

$$\Phi = \int_0^h \int_0^{r\pm\sqrt{R^2-z^2}} \int_0^{2\pi} z r d\phi dr dz \Rightarrow$$

$$\Phi = 2\pi \int_0^h \int_0^{r\pm\sqrt{R^2-z^2}} z r dr dz \Rightarrow$$

$$\Phi = 2\pi \int_0^h z \left[\frac{r^2}{2}\right]_0^{r\pm\sqrt{R^2-z^2}} dz \Rightarrow$$

$$\Phi = \pi \int_0^h z\left(\gamma \pm \sqrt{R^2-z^2}\right)^2 dz \Rightarrow$$

$$\Phi = \pi \int_0^h z\left(R^2 - z^2 + \gamma^2 \pm 2\gamma\sqrt{R^2-z^2}\right) dz \Rightarrow$$

$$\Phi = \pi \int_0^h \left(\left(R^2+\gamma^2\right)z - z^3 \pm 2\gamma z\sqrt{R^2-z^2}\right) dz \Rightarrow$$

$$\Phi = \pi\left[\frac{1}{2}\left(R^2+\gamma^2\right)z^2 - \frac{z^4}{4} \pm 2\gamma\left(-\frac{\sqrt{\left(R^2-z^2\right)^3}}{3}\right)\right]_0^h \Rightarrow$$

$$\Phi = \pi\left(\frac{1}{2}\left(R^2+(r\mp R)^2\right)h^2 - \frac{h^4}{4} \pm \frac{2}{3}(r\mp R)\left(\sqrt{\left(R^2\right)^3} - \sqrt{\left(R^2-h^2\right)^3}\right)\right) \Rightarrow$$

$$\Phi = \pi R^4\left(\frac{1}{2}\left(1+\left(\frac{r}{R}\mp 1\right)^2\right)\left(\frac{h}{R}\right)^2 - \frac{1}{4}\left(\frac{h}{R}\right)^4 \pm \frac{2}{3}\left(\frac{r}{R}\mp 1\right)\left(1-\sqrt{\left(1-\left(\frac{h}{R}\right)^2\right)^3}\right)\right) \Rightarrow$$

$$\Phi = \pi R^4\left(\frac{1}{2}\left(1+\left(1\mp\frac{r}{R}\right)^2\right)\left(\frac{h}{R}\right)^2 - \frac{1}{4}\left(\frac{h}{R}\right)^4 - \frac{2}{3}\left(1\mp\frac{r}{R}\right)\left(1-\sqrt{\left(1-\left(\frac{h}{R}\right)^2\right)^3}\right)\right) \Rightarrow$$

$$\Phi = \pi R^4\left(\frac{1}{2}\left(1+\mu^2\right)\lambda^2 - \frac{\lambda^4}{4} - \frac{2\mu}{3}\left(1-\sqrt{\left(1-\lambda^2\right)^3}\right)\right)$$

hvor

$$\mu = 1 \mp \frac{r}{R} \quad , \quad \begin{cases} - \text{ i tilfælde I}, & 0 \le t < r \\ + \text{ i tilfælde II}, & 0 \le r < t \end{cases}$$

$$\lambda = \frac{h}{R}$$

Massemidtpunktets z-koordinat er hermed

$$\Psi = R\,\frac{\frac{1}{2}\left(1+\mu^2\right)\lambda^2 - \frac{\lambda^4}{4} - \frac{2\mu}{3}\left(1-\sqrt{\left(1-\lambda^2\right)^3}\right)}{\lambda\left(1+\mu^2-\frac{\lambda^2}{3}-\mu\sqrt{1-\lambda^2}\right)-\mu\,\text{Arcsin}\,\lambda}$$

hvor

$$\mu = 1 \mp \frac{r}{R} \quad , \quad \begin{cases} - \text{ i tilfælde I}, & 0 \le t < r \\ + \text{ i tilfælde II}, & 0 \le r < t \end{cases}$$

$$\lambda = \frac{h}{R}$$

Inertimomenterne

Generel kan inertimomentet skrives som

$$I = \int \rho(x,y,x)\left[R(x,y,z)\right]^2 dV$$

Er massetætheden °*(x,y,z)*konstant kan den sættes udenfor integralet. Inertimomenterne med hensyn til koordinatakserne er nu

$$I_x = \rho\int\left(y^2+z^2\right)dV$$

$$I_y = \rho\int\left(x^2+z^2\right)dV$$

$$I_z = \rho\int\left(x^2+y^2\right)dV$$

Da der er symmetri omkring z-aksen har vi at

$$I_x = I_y$$

Inertimomentet om z-aksen findes ved følgende omformning

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$
$$dV = r\,d\phi\,dr\,dz$$

hvor

$$r^2 = x^2 + y^2$$

Vi får nu

$$I_z = \rho\int_0^k \int_0^{r\pm\sqrt{R^2-z^2}} \int_0^{2\pi} r^3\,d\phi\,dr\,dz \Rightarrow$$

$$I_z = 2\pi\rho\int_0^k \int_0^{r\pm\sqrt{R^2-z^2}} r^3\,dr\,dz \Rightarrow$$

$$I_z = 2\pi\rho\int_0^k \left[\frac{1}{4}r^4\right]_0^{r\pm\sqrt{R^2-z^2}} dz \Rightarrow$$

$$I_z = \frac{\pi}{2}\rho\int_0^k \left(\gamma \pm\sqrt{R^2-z^2}\right)^4 dz \Rightarrow$$

$$I_z = \frac{\pi}{2}\rho\int_0^k \left(\left(R^2-z^2\right)+\gamma^2 \pm 2\gamma\sqrt{R^2-z^2}\right)^2 dz \Rightarrow$$

$$I_z = \frac{\pi}{2}\rho\int_0^k \Big(\left(R^2-z^2\right)^2+\gamma^4+4\gamma^2\left(R^2-z^2\right)+2\gamma^2\left(R^2-z^2\right)$$
$$\pm 4\gamma^3\sqrt{R^2-z^2} \pm 4\gamma\sqrt{\left(R^2-z^2\right)^3}\Big)dz \Rightarrow$$

$$I_z = \frac{\pi}{2}\rho\int_0^k \Big(\left(R^2-z^2\right)^2+\gamma^4+6\gamma^2\left(R^2-z^2\right)$$
$$\pm 4\gamma^3\sqrt{R^2-z^2} \pm 4\gamma\sqrt{\left(R^2-z^2\right)^3}\Big)dz \Rightarrow$$

$$I_z = \frac{\pi}{2}\rho\Bigg[R^4z-\frac{2}{3}R^2z^3+\frac{1}{5}z^5+\gamma^4z+6\gamma^2\left(R^2z-\frac{1}{3}z^3\right)$$
$$\pm 4\gamma^3\left(\frac{z}{2}\sqrt{R^2-z^2}+\frac{R^2}{2}\operatorname{Arcsin}\frac{z}{R}\right)$$
$$\pm 4\gamma\left(\frac{z}{4}\sqrt{\left(R^2-z^2\right)^3}+\frac{3R^2z}{8}\sqrt{R^2-z^2}+\frac{3R^4}{8}\operatorname{Arcsin}\frac{z}{R}\right)\Bigg]_0^k \Rightarrow$$

$$I_z = \frac{\pi}{2}\rho\left[R^4 h - \frac{2}{3}R^2 h^3 + \frac{1}{5}h^5 + \gamma^4 h + 6\gamma^2\left(R^2 h - \frac{1}{3}h^3\right)\right.$$
$$\pm 4\gamma^3\left(\frac{h}{2}\sqrt{R^2 - h^2} + \frac{R^2}{2}\,\mathrm{Arcsin}\,\frac{h}{R}\right)$$
$$\left.\pm 4\gamma\left(\frac{h}{4}\sqrt{\left(R^2 - h^2\right)^3} + \frac{3R^2 h}{8}\sqrt{R^2 - h^2} + \frac{3R^4}{8}\,\mathrm{Arcsin}\,\frac{h}{R}\right)\right] \Rightarrow$$

$$I_z = \frac{\pi}{2}\rho\left[R^5\left(\frac{h}{R} - \frac{2}{3}\frac{h^3}{R^3} + \frac{1}{5}\frac{h^5}{R^5}\right) + 6\gamma^2 R^3\left(\frac{h}{R} - \frac{1}{3}\frac{h^3}{R^3}\right) + \gamma^4 R\frac{h}{R}\right.$$
$$\pm 4\gamma^3 R^2\left(\frac{h}{2R}\sqrt{1 - \left(\frac{h}{R}\right)^2} + \frac{1}{2}\,\mathrm{Arcsin}\,\frac{h}{R}\right)$$
$$\left.\pm 4\gamma R^4\left(\frac{h}{4R}\sqrt{\left(1 - \left(\frac{h}{R}\right)^2\right)^3} + \frac{3h}{8R}\sqrt{1 - \left(\frac{h}{R}\right)^2} + \frac{3}{8}\,\mathrm{Arcsin}\,\frac{h}{R}\right)\right] \Rightarrow$$

$$I_z = \frac{\pi}{2}\rho\left[R^5\lambda\left(1 - \frac{2}{3}\lambda^2 + \frac{1}{5}\lambda^4\right) + 6\gamma^2 R^3\lambda\left(1 - \frac{1}{3}\lambda^2\right) + \gamma^4 R\lambda\right.$$
$$\pm 4\gamma^3 R^2\left(\frac{\lambda}{2}\sqrt{1-\lambda^2} + \frac{1}{2}\,\mathrm{Arcsin}\,\lambda\right)$$
$$\left.\pm 4\gamma R^4\left(\frac{\lambda}{4}\sqrt{1-\lambda^2}\left(1-\lambda^2\right) + \frac{3\lambda}{8}\sqrt{1-\lambda^2} + \frac{3}{8}\,\mathrm{Arcsin}\,\lambda\right)\right] \Rightarrow$$

$$I_z = \frac{\pi}{2}\rho R^5\left[\lambda\left(1 - \frac{2}{3}\lambda^2 + \frac{1}{5}\lambda^4\right) + 6\lambda\left(\frac{\gamma}{R}\right)^2\left(1 - \frac{1}{3}\lambda^2\right) + \lambda\left(\frac{\gamma}{R}\right)^4\right.$$
$$\pm 2\left(\frac{\gamma}{R}\right)^3\left(\lambda\sqrt{1-\lambda^2} + \mathrm{Arcsin}\,\lambda\right)$$
$$\left.\pm \frac{\gamma}{R}\left(\lambda\sqrt{1-\lambda^2}\left(\frac{3}{2} + \left(1-\lambda^2\right)\right) + \frac{3}{2}\,\mathrm{Arcsin}\,\lambda\right)\right] \Rightarrow$$

$$I_z = \frac{\pi}{2}\rho R^5\left[\lambda\left(1 - \frac{2}{3}\lambda^2 + \frac{1}{5}\lambda^4\right) + 6\lambda\left(\frac{r}{R} \mp 1\right)^2\left(1 - \frac{1}{3}\lambda^2\right) + \lambda\left(\frac{r}{R} \mp 1\right)^4\right.$$
$$\pm 2\left(\frac{r}{R} \mp 1\right)^3\left(\lambda\sqrt{1-\lambda^2} + \mathrm{Arcsin}\,\lambda\right)$$
$$\left.\pm \left(\frac{r}{R} \mp 1\right)\left(\lambda\sqrt{1-\lambda^2}\left(\frac{3}{2} + \left(1-\lambda^2\right)\right) + \frac{3}{2}\,\mathrm{Arcsin}\,\lambda\right)\right] \Rightarrow$$

$$I_z = \frac{\pi}{2}\rho R^5\left[\lambda\left(1-\frac{2}{3}\lambda^2+\frac{1}{5}\lambda^4\right)+6\lambda\left(1\mp\frac{r}{R}\right)^2\left(1-\frac{1}{3}\lambda^2\right)+\lambda\left(1\mp\frac{r}{R}\right)^4\right.$$
$$-2\left(1\mp\frac{r}{R}\right)^3\left(\lambda\sqrt{1-\lambda^2}+\operatorname{Arcsin}\lambda\right)$$
$$\left.-\left(1\mp\frac{r}{R}\right)\left(\lambda\sqrt{1-\lambda^2}\left(\frac{3}{2}+\left(1-\lambda^2\right)\right)+\frac{3}{2}\operatorname{Arcsin}\lambda\right)\right]\Rightarrow$$

$$I_z = \frac{\pi}{2}\rho R^5\left[\lambda\left(1-\frac{2}{3}\lambda^2+\frac{1}{5}\lambda^4\right)+6\lambda\mu^2\left(1-\frac{1}{3}\lambda^2\right)+\lambda\mu^4\right.$$
$$-2\mu^3\left(\lambda\sqrt{1-\lambda^2}+\operatorname{Arcsin}\lambda\right)$$
$$\left.-\mu\left(\lambda\sqrt{1-\lambda^2}\left(\frac{3}{2}+\left(1-\lambda^2\right)\right)+\frac{3}{2}\operatorname{Arcsin}\lambda\right)\right]\Rightarrow$$

$$I_z = \frac{\pi}{2}\rho R^5\left[\lambda\left(\mu^4+6\mu^2+1\right)-\lambda^3\left(2\mu^2+\frac{2}{3}\right)+\frac{1}{5}\lambda^5\right.$$
$$\left.-\mu\left(\frac{3}{2}+2\mu^2\right)\operatorname{Arcsin}\lambda-\lambda\mu\sqrt{1-\lambda^2}\left(2\mu^2-\lambda^2+\frac{5}{2}\right)\right]$$

hvor

$$\mu = 1\mp\frac{r}{R}\quad,\quad\begin{cases}-\ \text{i tilfælde I},\ \ 0\le t<r\\ +\ \text{i tilfælde II},\ \ 0\le r<t\end{cases}$$

$$\lambda = \frac{h}{R}$$

Inertimomentet om henholdsvis x- og y-aksen findes nu ved analog omformning. Vi får

$$I_x = I_y = \rho\int_0^h\int_0^{r\pm\sqrt{R^2-z^2}}\int_0^{2\pi}\left(r^2\cos^2\varphi+z^2\right)r\,d\phi\,dr\,dz\Rightarrow$$

$$I_x = I_y = \rho\int_0^h\int_0^{r\pm\sqrt{R^2-z^2}}\left[r^3\left(\frac{\varphi}{2}+\frac{\sin 2\varphi}{4}\right)+z^2r\varphi\right]_0^{2\pi}dr\,dz\Rightarrow$$

$$I_x = I_y = \pi\rho\int_0^h\int_0^{r\pm\sqrt{R^2-z^2}}\left[r^3+2z^2r\right]dr\,dz\Rightarrow$$

$$I_x = I_y = \pi\rho\int_0^h\left[\frac{r^4}{4}+z^2r^2\right]_0^{r\pm\sqrt{R^2-z^2}}dz\Rightarrow$$

$$I_x = I_y = \pi\rho\int_0^h\left[\frac{1}{4}\left(\gamma\pm\sqrt{R^2-z^2}\right)^4+z^2\left(\gamma\pm\sqrt{R^2-z^2}\right)^2\right]dz\Rightarrow$$

$$I_x = I_y = \frac{1}{2}I_z+\pi\rho\int_0^h z^2\left(\left(R^2-z^2\right)+\gamma^2\pm2\gamma\sqrt{R^2-z^2}\right)dz\Rightarrow$$

$$I_x = I_y = \frac{1}{2} I_z + \pi\rho \int_0^h \left(\left(R^2 + \gamma^2\right) z^2 - z^2 \pm 2\gamma z^2 \sqrt{R^2 - z^2} \right) dz \Rightarrow$$

$$I_x = I_y = \frac{1}{2} I_z + \pi\rho \left[\frac{1}{3}\left(R^2 + \gamma^2\right) z^3 - \frac{1}{5} z^5 \pm 2\gamma \left(-\frac{z}{4}\sqrt{\left(R^2 - z^2\right)^3} + \frac{zR^2}{8}\sqrt{R^2 - z^2} + \frac{R^4}{8} \operatorname{Arcsin} \frac{z}{R} \right) \right]_0^h \Rightarrow$$

$$I_x = I_y = \frac{1}{2} I_z + \pi\rho \left[\frac{1}{3}\left(R^2 + \gamma^2\right) h^3 - \frac{1}{5} h^5 \pm 2\gamma \left(-\frac{h}{4}\sqrt{\left(R^2 - h^2\right)^3} + \frac{hR^2}{8}\sqrt{R^2 - h^2} + \frac{R^4}{8} \operatorname{Arcsin} \frac{h}{R} \right) \right] \Rightarrow$$

$$I_x = I_y = \frac{1}{2} I_z + \pi\rho R^5 \left[\frac{1}{3}\left(1 + \left(\frac{\gamma}{R}\right)^2\right)\left(\frac{h}{R}\right)^3 - \frac{1}{5}\left(\frac{h}{R}\right)^5 \pm 2\frac{\gamma}{R} \left(-\frac{h}{4R}\sqrt{\left(1 - \left(\frac{h}{R}\right)^2\right)^3} + \frac{h}{8R}\sqrt{1 - \left(\frac{h}{R}\right)^2} + \frac{1}{8} \operatorname{Arcsin} \frac{h}{R} \right) \right] \Rightarrow$$

$$I_x = I_y = \frac{1}{2} I_z + \pi\rho R^5 \left[\frac{1}{3}\left(1 + \mu^2\right)\lambda^3 - \frac{1}{5}\lambda^5 - 2\mu \left(-\frac{\lambda}{4}\sqrt{\left(1 - \lambda^2\right)^3} + \frac{\lambda}{8}\sqrt{1 - \lambda^2} + \frac{1}{8} \operatorname{Arcsin} \lambda \right) \right] \Rightarrow$$

$$I_x = I_y = \frac{1}{2} I_z + \frac{\pi}{4}\rho R^5 \left[\frac{4}{3}\lambda^3 - \frac{4}{3}\mu^2\lambda^3 - \frac{4}{5}\lambda^5 + 2\lambda\mu\sqrt{\left(1 - \lambda^2\right)^3} - \lambda\mu\sqrt{1 - \lambda^2} - \mu \operatorname{Arcsin} \lambda \right] \Rightarrow$$

$$I_x = I_y = \frac{1}{2} I_z + \frac{\pi}{4}\rho R^5 \left[\frac{4}{3}\lambda^3 - \frac{4}{3}\mu^2\lambda^3 - \frac{4}{5}\lambda^5 + \lambda\mu\sqrt{1 - \lambda^2}\left(1 - 2\lambda^2\right) - \mu \operatorname{Arcsin} \lambda \right]$$

Den udledte formel for I_z indsættes nu i udtrykket

$$I_x = I_y = \frac{\pi}{4}\rho R^5 \left[\lambda\left(\mu^4 + 6\mu^2 + 1\right) - \lambda^3\left(2\mu^2 + \frac{2}{3} - \frac{4}{3} + \frac{4}{3}\mu^2\right) - \lambda^5\left(\frac{4}{5} - \frac{1}{5}\right) - \lambda\mu\sqrt{1 - \lambda^2}\left(2\mu^2 - \lambda^2 + \frac{5}{2} - 1 + 2\lambda^2\right) - \mu\left(\frac{3}{2} + 2\mu^2 + 1\right) \operatorname{Arcsin} \lambda \right] \Rightarrow$$

$$I_x = I_y = \frac{\pi}{4}\rho R^5 \left[\lambda\left(\mu^4 + 6\mu^2 + 1\right) - \frac{2\lambda^3}{3}\left(5\mu^2 - 1\right) - \frac{3\lambda^5}{5} - \lambda\mu\sqrt{1 - \lambda^2}\left(2\mu^2 + \lambda^2 + \frac{3}{2}\right) - \mu\left(\frac{5}{2} + 2\mu^2\right) \operatorname{Arcsin} \lambda \right] \Rightarrow$$

hvor

$$\mu = 1 \mp \frac{r}{R} \quad , \quad \begin{cases} - \text{ i tilfælde I}, & 0 \le t < r \\ + \text{ i tilfælde II}, & 0 \le r < t \end{cases}$$

$$\lambda = \frac{h}{R}$$

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